Concept: Lagrange multipliers for constrained optimization.

Consider: $\begin{cases} \text{minimize/maximize } f(x) \\ \text{subject to the constraint } g(x) = 0 \end{cases}$

If x is a local minimum or maximum of 'f' subject to constraint 'g' $= 0$, and if $\nabla g(x) \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that the following system of equations are satisfied:

$$\left.\begin{aligned} \nabla f(x) + \lambda \nabla g(x) &= 0, \text{ -(a)} \\ g(x) &= 0 \text{ -(b)} \end{aligned}\right\} \text{ — (1)}$$

where ∇ denotes gradient.

Example: $f(p,q,r) = \frac{2n^3}{pqr}$

$g(p,q,r) = \frac{n^2}{qr} + \frac{n^2}{qp} + \frac{n^2}{pr} - M \leq 0$

we will consider the equality case i.e.

$\frac{n^2}{qr} + \frac{n^2}{qp} + \frac{n^2}{pr} - M = 0$

$$\nabla f(p,q,r) = \left(\frac{\partial}{\partial p}\left(\frac{2n^3}{pqr}\right), \frac{\partial}{\partial q}\left(\frac{2n^3}{pqr}\right), \frac{\partial}{\partial r}\left(\frac{2n^3}{pqr}\right) \right)$$

$$= \left(\frac{2n^3}{qr} \cdot \frac{-1}{p^2}, \frac{2n^3}{pr} \cdot \frac{-1}{q^2}, \frac{2n^3}{pq} \cdot \frac{-1}{r^2} \right)$$

$$\nabla g(p,q,r) = \left(\frac{-n^2}{p^2}\left(\frac{1}{q}+\frac{1}{r}\right), \frac{-n^2}{q^2}\left(\frac{1}{p}+\frac{1}{r}\right), \frac{-n^2}{r^2}\left(\frac{1}{p}+\frac{1}{q}\right) \right)$$

Substituting ∇f and ∇g in (1) we get

$\lambda = \frac{-2n}{q+r} \Rightarrow \boxed{p = q = r}$ (when you substitute for λ in sub equations of (1).(a))

Now, substituting $p = q = r = x$ in $g(p,q,r)$ $\frac{3n^2}{x^2} - M = 0$

$\Rightarrow x = \sqrt{\frac{3n^2}{M}} \Rightarrow$ each sub-matrix is of size $\frac{n}{\sqrt{\frac{3n^2}{M}}} \times \frac{n}{\sqrt{\frac{3n^2}{M}}} = \boxed{\frac{M}{3}}$