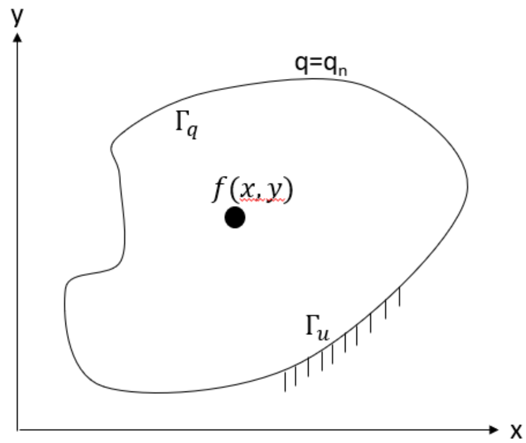
CS601, Lecture 31/10/2022 and Half of 3/11/2022 – Finite Element Method contd. (2D problems)

The main topic of today's class is 2D steady state diffusion problem. The problem is as illustrated below:



- Γ_u= Part of the boundary, where boundary conditions are specified. Here, $T_{\Gamma_u} = \tilde{T}$, meaning the temperature is known at boundary Γ_u.
- Γ_q= Part of the boundary, where *flux* is specified.
- Flux = q = q_n =incoming or outgoing heat energy
- $f(x,y)$ = Heat source
- Note: $\Gamma = \Gamma_u + \Gamma_q$, represents entire boundary.

Problem: *find the temperature T at any point (x,y) over the domain.*

The steady state heat diffusion equation is given by:

$$\mathrm{K}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+f(x,y) = 0 \qquad \text{———} \quad (1)$$

When $f(x,y) = 0$, the above equation becomes $\mathrm{K}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) = 0$ and is also called a Laplace's equation.

We also know that at some boundary Γ_u :

$$T_{\Gamma_u} = \tilde{T} \qquad \text{———} \quad (2)$$

And the Neumann B.C. along boundary Γ_q ($\hat{n}_x$ and $\hat{n}_y$ are unit vectors along x and y direction resp.)

$$\mathrm{K}\left(\frac{\partial T}{\partial x}\hat{n}_x+\frac{\partial T}{\partial y}\hat{n}_y\right) = q_n \qquad \text{———} \quad (3)$$

(1), (2), and (3) represent the strong-form of the steady-state 2D heat diffusion problem.
The first step in the FEM approach is to transform the strong-form to weak-form. This is done by integrating the product of the weight function and the residual over the domain and equating to zero i.e.

$$\int_\Omega \omega\, R = 0$$

$$= \int_\Omega \omega\left(\mathrm{K}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+f(x,y)\right) = 0 \quad \text{—} \quad (4)$$

Road to obtaining weak-form:

We know that:

$$\frac{\partial}{\partial x}\left[\omega \mathrm{K}\frac{\partial T}{\partial x}\right] = \mathrm{K}\frac{\partial \omega}{\partial x}\frac{\partial T}{\partial x} + \omega \mathrm{K}\ \frac{\partial^2 \mathrm{T}}{\partial x^2}$$
$$\frac{\partial}{\partial y}\left[\omega \mathrm{K}\frac{\partial T}{\partial y}\right] = \mathrm{K}\frac{\partial \omega}{\partial y}\frac{\partial T}{\partial y} + \omega \mathrm{K}\ \frac{\partial^2 \mathrm{T}}{\partial y^2}$$

Substituting for the second-order partial derivative term in (4):

$$-\int_\Omega \mathrm{K}\frac{\partial \omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\,\mathrm{d}\Omega + \int_\Omega \frac{\partial}{\partial \mathrm{x}}\left(\omega \mathrm{K}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right)\mathrm{d}\Omega - \int_\Omega \mathrm{K}\frac{\partial \omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\,\mathrm{d}\Omega + \int_\Omega \frac{\partial}{\partial \mathrm{y}}\left(\omega \mathrm{K}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right)\mathrm{d}\Omega + \int_\Omega \omega f \mathrm{d}\Omega = 0 \qquad \text{—— (5)}$$

A little bit of background (on Gauss Divergence Theorem):

- A function that takes in e.g. two variables (x,y in 2D) and outputs *one value* given by $f(x,y)$ is called a *scalar-valued function*.
- A function that takes in e.g. two variables (x,y in 2D) and outputs *a vector* in (x,y), i.e. the value of $f(x,y)$ is a vector, is called a *vector-valued function*.
- Suppose we define a function that outputs a vector $\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix}$, where the vector components are partial derivatives of the function f, we call such a vector-valued function the *gradient* or ∇ (nabla). $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\hat{n}_x + \frac{\partial f}{\partial y}\hat{n}_y$, where $\hat{n}_x$ and $\hat{n}_y$ are unit vectors along x and y direction resp.
- Divergence of a vector field $\nabla\ .\ (f_x, f_y)$ is a scalar-valued function $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$
- Let a be a vector field. Then, the Gauss divergence theorem relates the surface integral (2D) to boundary/contour/line integral (1D) through the divergence operator:

$$\int_\Omega \nabla . a\ d\Omega = \int_\Gamma a_i\ \hat{n}_i d\Gamma$$

Applying Gauss divergence theorem in (5):

$$
\begin{aligned}
-\int_{\Omega} & \mathrm{K}\frac{\partial \omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\, d\Omega \\
& + \int_{\Omega} \frac{\partial}{\partial \mathrm{x}}\left(\omega \mathrm{K}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right) d\Omega \quad \leftarrow \quad \int_{\Gamma} \mathrm{K}\omega\frac{\partial T}{\partial x}\, \hat{n}_x d\Gamma \\
& - \int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\, d\Omega \\
& + \int_{\Omega} \frac{\partial}{\partial \mathrm{y}}\left(\omega \mathrm{K}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right) d\Omega \quad \leftarrow \quad \int_{\Gamma} \mathrm{K}\omega\frac{\partial T}{\partial y}\, \hat{n}_y d\Gamma \\
& + \int_{\Omega} \omega f\, d\Omega = 0
\end{aligned}
$$

$$
= \int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\, d\Omega + \int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\, d\Omega = \int_{\Gamma} \mathrm{K}\omega\left[\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\hat{n}_y + \frac{\partial \mathrm{T}}{\partial \mathrm{x}}\hat{n}_x\right] d\Gamma + \int_{\Omega} \omega f\, d\Omega
$$

$$
= \int_{\Omega} \mathrm{K}\left(\frac{\partial \omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}} + \frac{\partial \omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right) d\Omega = \int_{\Gamma} \mathrm{K}\omega\left[\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\hat{n}_y + \frac{\partial \mathrm{T}}{\partial \mathrm{x}}\hat{n}_x\right] d\Gamma + \int_{\Omega} \omega f\, d\Omega \quad \text{—— (6)}
$$

Stiffness matrix

Boundary condition term (vector) contains coefficient of weight function

$\mathrm{K}\left[\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\hat{n}_y + \frac{\partial \mathrm{T}}{\partial \mathrm{x}}\hat{n}_x\right]$

Force vector

The stiffness matrix is identified by the bilinear term $\frac{\partial \omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}$ or $\frac{\partial \omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}$

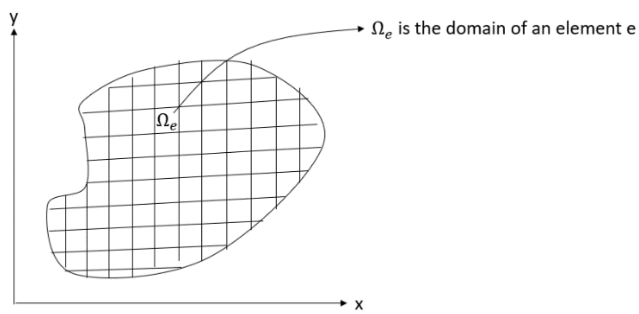
(6) is the weak-form equation. Note that this is the general equation. In summary, we have started with governing equations which are in the strong-form. Next, we used chain-rule / integration by parts to reduce the degree of the equation e.g. to substituting second-order with first-order derivative terms. Next, we find the boundary term, primary, and secondary variables. To do this, for the 2D problem, we apply the Gauss-divergence theorem to convert the domain or surface integral to boundary integral to obtain the weak-form equation.

Next, we write elemental equations. To do this, first discretize the domain into N-sided elements, where each element is assumed to have N nodes. After elemental equations for all the elements of the domain are written, perform assembly to construct the global stiffness matrix, boundary term vector, and force vector. Note that the elements of the Global stiffness matrix and force vector involve computation of integrals over limits in the physical domain. Analytically solving the integrals is too difficult or not possible often. Hence, to numerically solve the integrals, Gauss-Quadrature method is used. The details of this method were discussed in the last class.

The equation written in (6) has the symbolic weight function (ω) for a general n-sided element. These symbolic weight functions are to be replaced by the shape functions (N_i)s that were derived for triangular elements. Thus, we obtained 3 equations per element consisting of 3 bilinear integral terms on the LHS. In other words, the elemental stiffness matrix was 3x3.

Then, the assembly is done combining such 3x3 matrices. While doing this, we should consider constraints / requirements to be met while ensuring the continuity of the solution at nodes common to elements.

Before we write the elemental equations, we need to discretize the domain. The picture illustrates an example:



Writing the weak-form equation of (6) for the element's domain Ω_e

$$\int_{\Omega^e} K\left(\frac{\partial\omega}{\partial x}\frac{\partial T^e}{\partial x} + \frac{\partial\omega}{\partial y}\frac{\partial T^e}{\partial y}\right) d\Omega^e = \int_{\Gamma^e} K\omega\left[\frac{\partial T^e}{\partial y}\hat{n}_y + \frac{\partial T^e}{\partial x}\hat{n}_x\right] d\Gamma^e + \int_{\Omega^e} \omega f d\Omega^e$$

Where, T^e denotes the approximate solution (temperature) within the element's domain, and Γ^e denotes element's boundary.

The above equation can also be written as:

$$\int_{\Omega^e} K\left(\frac{\partial\omega}{\partial x}\frac{\partial T^e}{\partial x} + \frac{\partial\omega}{\partial y}\frac{\partial T^e}{\partial y}\right) d\Omega^e = \int_{\Gamma^e} \omega\hat{q}^n d\Gamma^e + \int_{\Omega^e} \omega f d\Omega^e \quad \text{——} \quad (7)$$

Where, $\hat{q}^n$ denotes the flux for the element. Note that this flux is different from the flux q^n specified for the entire domain.

Also, T^e can be written as: $\sum_{i=1}^{n} N_i T_i^e$

- where, T_i^e is the temperature at the i^{th} node of the element (Similar to what we did in class on 5/10 when we approximated the displacement, $u(x)$, between the nodes of an element in terms of nodal displacements using linear functions: $\tilde{u}(x) = N_1(x)u_1 + N_2(x)u_2)$

- and N_i are shape functions (now of two variabes $N_i(x,y)$).

As per the Galerkin approach, ω can be replaced with N_i. Rewriting (7):

$$\int_{\Omega^e} K\left(\frac{\partial N_i}{\partial x}\frac{\partial T^e}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial T^e}{\partial y}\right) d\Omega^e = \int_{\Gamma^e} N_i \hat{q}^n d\Gamma^e + \int_{\Omega^e} N_i f d\Omega^e$$

Substituting for $T^e = \sum_{i=1}^{n} N_i T_i^e$ the above equation:

$$\int_{\Omega^e} K\left(\frac{\partial N_i}{\partial x}\frac{\partial}{\partial x}\sum_{j=1}^{n} N_j T_j^e + \frac{\partial N_i}{\partial y}\frac{\partial}{\partial y}\sum_{j=1}^{n} N_j T_j^e\right) d\Omega^e = \int_{\Gamma^e} N_i \hat{q}^n d\Gamma^e + \int_{\Omega^e} N_i f d\Omega^e$$

The summation symbol can be omitted (as per the conventional notations) to represent the above equation as:

$$\int_{\Omega^e} K\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right) T_j^e d\Omega^e = \int_{\Gamma^e} N_i \hat{q}^n d\Gamma^e + \int_{\Omega^e} N_i f d\Omega^e$$

The T_j^e term can be factored out because the nodal temperature is assumed to be a constant (unknown).

The above equation is sometimes compactly written as:

$$K_{ij} T_j^e = q_i + f_i \qquad (8)$$

where, $K_{ij} = \int_{\Omega^e} \mathrm{K} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} d\Omega^e$

Equation (3) needs to be written for all nodes of an element and the exercise needs to be repeated for all elements of the domain. The matrices obtained w.r.t. an element need to be assembled afterwards.

For an n-sided element:

$$T = N_1 T_1 + N_2 T_2 + ... + N_n T_n$$

Hence,

$$\frac{\partial T}{\partial x} = \frac{\partial N_1}{\partial x} T_1 + \frac{\partial N_2}{\partial x} T_2 + ... + \frac{\partial N_n}{\partial x} T_n$$
$$\frac{\partial T}{\partial y} = \frac{\partial N_1}{\partial y} T_1 + \frac{\partial N_2}{\partial y} T_2 + ... + \frac{\partial N_n}{\partial y} T_n$$

Rewriting in Ax=b form:

$$\begin{bmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$$

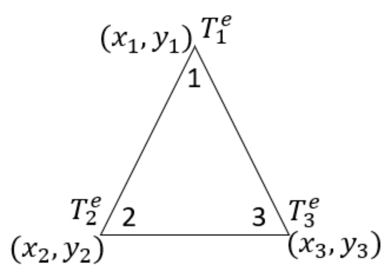
Call this matrix as B

Define a matrix C as: $\begin{bmatrix} K & 0 \\ 0 & K \end{bmatrix}$ then,

$$K_{ij} = \int_{\Omega^e} B^T C B \, d\Omega^e$$

where, B^T denotes transpose of the matrix B.

Deriving shape functions for a triangular element, n=3 (n=3 sides):



$$\begin{aligned} T &= C_1 + C_2 x + C_3 y \\ T_1^e &= C_1 + C_2 x_1 + C_3 y_1 \\ T_2^e &= C_1 + C_2 x_2 + C_3 y_2 \\ T_3^e &= C_1 + C_2 x_3 + C_3 y_3 \end{aligned}$$

Solving for C_1 , C_2, and C_3 (in terms of x and y), and substituting in $T = C_1 + C_2 x + C_3 y$ we get

T = ( ) T_1^e + ( ) T_2^e + ( ) T_3^e

Denotes: N_1 N_2 N_3

$$N_1 = \frac{1}{2A^e}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$N_2 = \frac{1}{2A^e}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

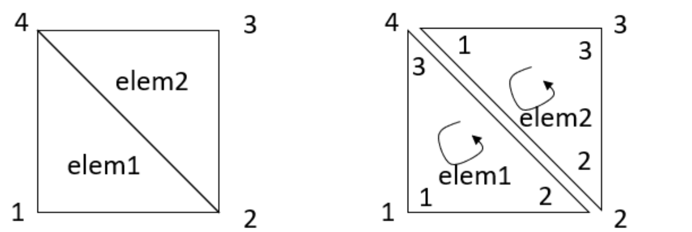
$$N_3 = \frac{1}{2A^e}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

Where, A^e is the area of the triangular element.

For the above N_i's, the elements of the B matrix are constants (i.e. not a function of x or y because N_is are linear functions of x or y and the partial derivatives of N_is give out constants.) Such a constant matrix is called in the literature as "Constant strain triangular element".

---------------------------This is not discussed in class. This is for your information-----------

Once such matrix is obtained for all elements, we do the assembly. While doing assembly, first number the nodes in anti-clockwise in local node numbering as shown:



Global numbering of nodes

Local numbering of nodes

To maintain continuity of the solution obtained (i.e. temperature at nodes common to elements must be same and the flux should be zero on the common edges – there are two common nodes 2 and 4):

$T_1 = T_1^{e_1}$ (meaning temperature at node 1 in global numbering = temperature at node 1 of element 1)

$$T_4 = T_3^{e_2}$$
$$T_2 = T_2^{e_1} = T_2^{e_2}$$
$$T_4 = T_3^{e_1} = T_1^{e_2}$$

Regrading flux term (q_i in Eqn. (8)):
At node number 2 (global numbering):
The outflux of node 2 in element 1 + outflux of node 2 in element 2 = 0

$$\int_{\Gamma^{2-3}} N_2 q_{2-3} d\Gamma^{2-3} + \int_{\Gamma^{1-2}} N_2 q_{1-2} d\Gamma^{1-2}$$

Similarly, at node number 4 (global numbering):
The outflux of node 3 in element 1 + outflux of node 1 in element 2 = 0

$$\int_{\Gamma^{2-3}} N_3 q_{2-3} d\Gamma^{2-3} + \int_{\Gamma^{1-2}} N_1 q_{1-2} d\Gamma^{1-2}$$