CS601, Lecture 27/10/2022 – Finite Element Method contd. (Numerical Integration)

The Ax=B form obtained in the last class involved computation of integrals over the domain (for computing the K matrix and for computing the body force vector term in the RHS). Sometimes, it is non-trivial to compute the integration. When E and A are functions of space variables (x), when higher-order polynomial shape functions are used to approximate the solution for better accuracy (e.g. with 3-node or 5-node elements we need higher-order polynomial functions.), we need to numerically compute the integral function. Also, the domain always consists of more than one element. In this scenario, the elemental stiffness matrices from many constituent elements are to be combined to form a global stiffness matrix.

A technique called Gauss-Quadrature, is used commonly in FEM to numerically integrate functions appearing as terms in the Ax=B form. In this approach:

1. The *physical domain* is mapped to the *natural domain* first.
2. Integrals are then approximated with summations using Gauss quadrature rules.

For the 1D rod element, the physical domain is $x_a$to x_b . The natural domain is always defined from -1 to 1. Gauss-Quadrature rules exist as 1-point, 2-point, ..., N-point rule etc. Depending on the polynomial appearing as the integral function, you use different rules. *Fact: using N-point quadrature rule, you can integrate (2N-1)-degree polynomial correctly.*

The basic requirement for the Gauss-Quadrature rules to be applied is that the integral limits range from -1 to 1. However, the limits in the integral terms appearing in the elemental equations in Ax=B form range from x_a to x_b. Hence, the integral functions need to be *mapped* from physical domain to the natural domain. E.g.

$$K_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx = \int_{x_a}^{x_b} F(x)dx \quad \Rightarrow \quad K_{ij} = \int_{-1}^{1} \tilde{F}(\xi)d\xi$$

$\int_{-1}^{1} \tilde{F}(\xi)d\xi$ is numerically computed as summation:

$\int_{-1}^{1} \tilde{F}(\xi)d\xi = \sum_{i=1}^{N} w_i \tilde{F}(\xi_i)$, where w_i is the quadrature weight of the i[th] quadrature point and $\xi_i$is the corresponding location. E.g.

In two-point quadrature $w_1\tilde{F}(\xi_1) + w_2\tilde{F}(\xi_2),\ w_1 = w_2 = 1, \xi_1 = -\frac{1}{\sqrt{3}}, \xi_2 = \frac{1}{\sqrt{3}}$.

In one-point quadrature, $w_1\tilde{F}(\xi_1), w_1 = 2, \xi_1 = 0$

Earlier, the displacement at x, a point between the nodes, was written in terms of the nodal displacements using linear functions N_1 and N_2 as follows:

$$\tilde{u}(x) = N_1(x)u_1 + N_2(x)u_2$$

where u_1 and u_2 are nodal displacements at nodes 1 and 2 resp.

Now, we would need the equivalent of:

$$\tilde{u}(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$$

To begin the mapping of the domain from physical (x) to natural (ξ), let us begin by writing x as:

$$x = a + b\xi$$

We know $\xi = -1$, when $x = x_a$ AND $\xi = 1$ when $x = x_b$. So, using these two conditions, we get two equations that we can solve to get a and b. Substituting the values of a and b in $x = a + b\xi$, we get:

$x = x_a \frac{1-\xi}{2} + x_b \frac{1+\xi}{2}$. So, we can write x as:

$x = x_a N_1(\xi) + x_b N_2(\xi)$, where $N_1 = \frac{1-\xi}{2}$ and $N_2 = \frac{1+\xi}{2}$ can be seen as linear functions of ξ.

In *isoparametric* FEM, we use the same functions N_1 and N_2 to approximate the solution $\tilde{u}(\xi)$ as:

$$\tilde{u}(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$$

Now, we need to express the N_i and N_j (which are linear functions of x) in $K_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx$ using $N_1(\xi)$ and $N_2(\xi)$. So,

$\frac{dN_i}{dx}$ needs to be converted in terms of $\frac{dN_i}{d\xi}$ and $\frac{d\xi}{dx}$ (chain rule).

We know that: $x = x_a N_1(\xi) + x_b N_2(\xi)$. OR $x = x_1 N_1 + x_2 N_2$ ($x_1 = x_a$and $x_2 = x_b$ are node 1 and node 2 coordinates resp.) OR $x = \sum_{k=1}^{2} x_k N_k$

$\frac{dx}{d\xi} = x_1 \frac{d}{d\xi} N_1 + x_2 \frac{d}{d\xi} N_2 = \sum_{k=1}^{2} x_k \frac{d}{d\xi} N_k$,

$\frac{dx}{d\xi} = J$, where J is called the Jacobian matrix (because the domain is 1D, we have a scalar value (only one component) for the Jacobian matrix).

$$K_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx = \int_{-1}^{1} EA \frac{dN_i}{d\xi}\frac{d\xi}{dx}\frac{dN_j}{d\xi}\frac{d\xi}{dx}\left(\sum_{k=1}^{2} x_k \frac{d}{d\xi} N_k\right) d\xi =$$

$$\int_{-1}^{1} EA \frac{dN_i}{d\xi}\frac{dN_j}{d\xi}[J]^{-1}[J]^{-1}[J] d\xi =$$

$$\int_{-1}^{1} EA \frac{dN_i}{d\xi}\frac{dN_j}{d\xi}[J]^{-1} d\xi$$

The integral function $EA \frac{dN_i}{d\xi}\frac{dN_j}{d\xi}[J]^{-1}$ above can be represented as $\tilde{F}(\xi)$

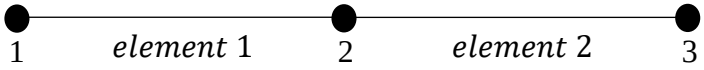
Using 1-point quadrature ($w_1 = 2, \xi_1 = 0$) and substituting for $\tilde{F}(0)$:
$\frac{dx}{d\xi} = x_1 \frac{d}{d\xi} N_1 + x_2 \frac{d}{d\xi} N_2 = x_1 \frac{d}{d\xi}\left(\frac{1-\xi}{2}\right) + x_2 \frac{d}{d\xi}\left(\frac{1+\xi}{2}\right) = \frac{-x_1}{2} + \frac{x_2}{2} = \frac{x_2 - x_1}{2} = \frac{L}{2}$

$$K_{11} = \int_{-1}^{1} EA \frac{dN_1}{d\xi}\frac{dN_1}{d\xi}[J]^{-1} d\xi = \frac{EA}{L}$$
$$K_{12} = \int_{-1}^{1} EA \frac{dN_1}{d\xi}\frac{dN_2}{d\xi}[J]^{-1} d\xi = -\frac{EA}{L}$$
$$K_{21} = \int_{-1}^{1} EA \frac{dN_2}{d\xi}\frac{dN_1}{d\xi}[J]^{-1} d\xi = -\frac{EA}{L}$$
$$K_{22} = \int_{-1}^{1} EA \frac{dN_2}{d\xi}\frac{dN_2}{d\xi}[J]^{-1} d\xi = \frac{EA}{L}$$

Thus, the elemental stiffness matrix is (when E and A are constants):

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now consider the rod with two elements. There are 3 nodes as shown below for the two elements:



For the 3 nodes, there are three degrees of freedom (i.e. 3 displacement values, each corresponding to a node). Since, there are 3 displacement values i.e. u^s , there would be 3 components to the x vector in the Ax=B form (Recall: x vector denotes the displacements). So, the *global stiffness matrix* combining the elemental matrices of elements 1 and 2 would be 3x3:

	Node: 1	2	3
Node: 1	$\frac{A_1E_1}{L_1}$	$-\frac{A_1E_1}{L_1}$	0
2	$-\frac{A_1E_1}{L_1}$	$\frac{A_1E_1}{L_1}+\frac{A_2E_2}{L_2}$	$-\frac{A_2E_2}{L_2}$
3	0	$-\frac{A_2E_2}{L_2}$	$\frac{A_2E_2}{L_2}$

, where $\frac{A_1E_1}{L_1}$ comes from elemental matrix for element 1 and $\frac{A_2E_2}{L_2}$ comes from elemental matrix for element 2. Note that the particular order of numbering nodes is important and dictates the values

of the entries in the global stiffness matrix. Based on the numbering followed the resultant stiffness matrix can be sparse (lot of zeros) or non-sparse. For a 1D problem with many elements and following the node numbering order as shown above, we get a tridiagonal matrix.