CS601, Lecture 20/10/2022 – Finite Element Method contd.

The previous lecture started with the FEM approach to solving PDEs. First, the method of weighted residuals was discussed and then the original PDE in *strong-form* was transformed to a *weak-form* equation. This lecture discusses applying the ideas discussed in the previous lecture to the FEM approach. First, the high-level steps of the FEM approach are listed, then the concept of element, shape functions, and the system of equations are discussed in the context of a 1D rod problem.

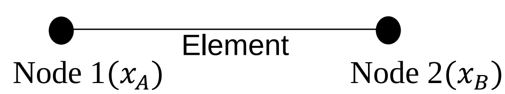
The steps in the FEM approach are:

1) construct the weak-form of the PDE
2) assume the form of approximate solution for a typical *element*. More on element in the next paragraph.
3) Derive finite element equations by substituting approximated solution in the weak form

An element is a resulting sub-domain obtained after the discretization step. Each element contains *nodes* (grid points in the FDM context. In FEM context, a node is not exactly a grid point. A grid point makes sense only in the FDM context while in the FEM context, an *element* is used. An element can contain nodes at the *boundaries or internally*. The number of nodes per element depends on the approximate solution considered. You will see this later in the example.

The approximate solution (in step 2 above) considered must satisfy the following properties:
- Continuous and differentiable
- Complete (e.g. $f(x) = c1 + c2x$ is a complete linear function. $g(x) = c1x + c2x^2$ is not a complete quadratic function.)

Consider a 2-node rod element:



x_A and x_B are the spatial coordinates. The element is of length $h^e = x_B - x_A$
Let the approximate solution for element e, u^e, be a linear function $f(x) = c1 + c2x$. So, the approximate solution at points x_A and x_B are given by the equations:

$$u_1^e = c1 + c2x_A$$
$$u_2^e = c1 + c2x_B$$

Writing in matrix form:

$$\begin{bmatrix} 1 & x_A \\ 1 & x_B \end{bmatrix} \begin{bmatrix} c1 \\ c2 \end{bmatrix} = \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix}$$

yields:

$$c1 = u_1^e - \frac{(u_2^e - u_1^e)}{x_B - x_A} x_A$$
$$c2 = \frac{(u_2^e - u_1^e)}{x_B - x_A}$$

Substituting for c1 and c2 in the approximated solution $c1 + c2x$:

$$c1 + c2x = \frac{u_1^e(x_B - x)}{(x_B - x_A)} + \frac{u_2^e(x - x_A)}{(x_B - x_A)}$$

Let us denote the approximated solution as $\tilde{u}$. Then $\tilde{u}$ =$N_1(x)u_1 + N_2(x)u_2$, where N_1, N_2, are the *shape functions* or *weight functions* or *interpolation functions*. The element is called as a 2-noded linear element.

Note that these are functions of x.

At point x_B, the value of function $N_1 = 0$ and point x_A the value of function $N_1 = 1$

At point x_B, the value of function $N_2 = 1$ and point x_A the value of function $N_2 = 0$

We denote $\tilde{u}$ as:

$$\tilde{u} \ = \ N_1u_1 + N_2u_2$$

or written in terms of longitudinal displacement:

$$\tilde{u}(x) \ = \ N_1(x)u_1 + N_2(x)u_2$$

u_1 and u_2 are displacements at the Nodes 1 and 2 resp. These are called *nodal / elemental displacements*.

Remark: we can choose an internal node in our element. In this case, we will have 3 nodes, and the approximated solution must be a quadratic when we have 3 nodes. For better accuracy, we can increase the polynomial degree (and also the internal nodes commensurately) or decrease the mesh size $(x_B - x_A)$. In practice, increasing the polynomial degree is more expensive to compute.

Recall the weak-form equation for the rod:

$$\left[\omega EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \ \omega \ F \ d\Omega = \int_\Omega \ EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx} \ d\Omega$$

Substituting for ω with N_1 and N_2. in the "weak-form" equation, we get the following two equations:

$$\left[N_1EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \ N_1 \ F \ d\Omega = \int_\Omega \ EA\frac{dN_1}{dx}\frac{d}{dx}(N_1u_1 + N_2u_2 \ ) \ d\Omega$$

$$\left[N_2EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \ N_2 \ F \ d\Omega = \int_\Omega \ EA\frac{dN_2}{dx}\frac{d}{dx}(N_1u_1 + N_2u_2 \ ) \ d\Omega$$

Considering the first equation, rewriting, and expanding (note the color coding is for readability only):

$$\left[N_1EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \ N_1 \ F \ d\Omega = \int_\Omega \ EA\frac{dN_1}{dx}\frac{d}{dx}(N_1u_1 + N_2u_2 \ ) \ d\Omega \qquad \text{(LHS=RHS)}$$

$$\Rightarrow \int_\Omega \ EA\frac{dN_1}{dx}\frac{d}{dx}(N_1u_1 + N_2u_2 \ ) \ d\Omega \ = \left[N_1EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \ N_1 \ F \ d\Omega \quad \text{(RHS=LHS)}$$

$$\Rightarrow \int_\Omega \ \mathrm{EA} \ \frac{dN_1}{dx}\frac{dN_1}{dx}\mathrm{u}_1 \ d\Omega \ + \ \int_\Omega \ \mathrm{EA} \ \frac{dN_1}{dx}\frac{dN_2}{dx}\mathrm{u}_2 \ d\Omega = \left[N_1EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \ N_1 \ F \ d\Omega$$

Similarly considering the second equation, rewriting, and expanding:

$$\int_\Omega \text{EA} \frac{dN_2}{dx}\frac{dN_1}{dx}\text{u}_1 \, d\Omega \, + \, \int_\Omega \text{EA} \frac{dN_2}{dx}\frac{dN_2}{dx}\text{u}_2 \, d\Omega = \left[N_2 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_2 \, F \, d\Omega$$

Using shorter notation on the LHS for the two equations expanded:

$$K_{11}u_1 + K_{12}u_2 = \left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_0^L N_1 \, F \, dx \qquad \text{———— (1)}$$

$$K_{21}u_1 + K_{22}u_2 = \left[N_2 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_0^L N_2 \, F \, dx \qquad \text{———— (2)}$$

Where, $K_{ij} = \int_0^L EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx$ and Ω ranges from 0 to L (dΩ becomes dx because of the domain is 1D).

----------------------------This was not discussed in the lecture but will follow.------------------

The Equations 1 and 2 can be expressed in Ax=B form (A is matrix, x is vector, and B is a vector) as:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_0^L N_1 \, F \, dx \\ \left[N_2 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_0^L N_2 \, F \, dx \end{bmatrix}$$

Stiffness matrix

displacements

Force vector