CS601, Lecture 18/10/2022 – Finite Element Method

This lecture begins with an introduction to an approach to solving PDES. This approach is called Finite Element Method (FEM).

Introduction: FDM is not practical/accurate approach to solve for all domains and all situations. Few reasons why FDM is not practical are as follows:

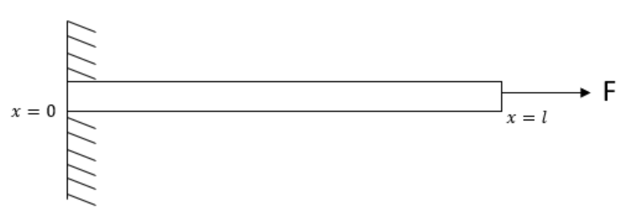
- Domain structure: FDM cannot handle accurately domains with irregular geometry/shapes. E.g., a domain with a curved boundary, domain with holes (with irregular boundaries).
- Truncation error: what is the guarantee that successive derivatives of the unknown function (e.g., u in $EA\frac{d^2u}{dx^2} + F)$ decrease in magnitude when we write the Taylor series approximation? for e.g., $f(x) = e^x$, the magnitude remains the same. As a result, when we ignore the higher order terms of the Taylor series and truncate the series, the numerical error / truncation error might be significant.
- Other physical properties: when there are additional mechanical engineering properties that need to be satisfied, FDM doesn't consider these.

FEM is a method of solving PDEs that overcomes the shortcomings of FDM to solve practical engineering problems in mechanical, electrical engineering, physics, aerospace engineering, among others.

Example: 1D structural problem

$$EA\frac{d^2u}{dx^2} + F = 0 \qquad \text{——— (1)}$$

The above equation is also called as equilibrium equation. Typically, F is given as bA, where b is the body force per unit area and A is the area. In this problem, the goal is to find stresses and strains at different points on the rod, which is fixed at one end and is subjected to a force at the other end. An illustration is shown below:



We have: Stress (σ) / Strain (ϵ) = E (Young's modulus)

Strain (ϵ) = change in length / original length = $\frac{\partial u}{dx}$

Substituting: $E\frac{\partial u}{dx} = \sigma$

To solve the PDE shown in (1), we need boundary conditions. Typical boundary conditions look like:

At $x = 0$, $u = 0$

At $x = L$, $u = EA\partial u/\partial x$ (the value is specified) ———— (2)

The PDE of equation (1) together with the boundary conditions in (2) is called as *strong-form of the PDE equation.* Suppose, $\tilde{u}$ is an approximate solution. Being approximate in nature, $\tilde{u}$ will not satisfy the equation (1) at all points x i.e.

$$EA\frac{d^2\tilde{u}}{dx^2} + F = R \qquad \text{———— (3)}$$

We say that $\tilde{u}$ has an associated error or *Residual* R. *The goal is to minimize the residual.* The method to minimize the residual is called as *weighted residual method.* This method simply multiplies the residual with a weighted function, takes the integral and equates to 0.

$$\int \omega\, R = 0$$

This approach is also called as Galerkin approach. ω are called as *weight functions* or *shape functions.* Substituting for R from (3) in the above equation,

$$\int \omega\,(EA\frac{d^2\,\tilde{u}}{dx^2} + F) = 0$$

$=\int_\Omega \omega\, EA\frac{d^2\,\tilde{u}}{dx^2}\, d\Omega + \int_\Omega \omega\, F\, d\Omega = 0$, where $\int_\Omega$ denotes integral over the domain Ω. For the 1D rod problem, this is integral over 0 to L.

We can substitute for $\omega\,\frac{d^2\,\tilde{u}}{dx^2}$ in the first term above with the following information from chain rule of differentiation:

$$\frac{d}{dx}\left[\omega\frac{d\tilde{u}}{dx}\right] = \frac{d\omega}{dx}\frac{d\tilde{u}}{dx} + \omega\,\frac{d^2\,\tilde{u}}{dx^2}$$

Substituting:

$$\int_\Omega \frac{d}{dx}(\omega EA\frac{d\tilde{u}}{dx})\, d\Omega - \int_\Omega EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}\, d\Omega + \int \omega\, F\, d\Omega = 0$$

Rearranging terms:

$$\left[\omega EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \omega\, F\, d\Omega = \int_\Omega EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}\, d\Omega \qquad \text{———— (4)}$$

In the above equation, $\left[\omega EA\frac{d\tilde{u}}{dx}\right]_0^L$ is called the *boundary term*. The coefficient of the weight function in the boundary term represents the Neumann BC. This coefficient is also called as *secondary variable*. $EA\frac{d\tilde{u}}{dx}$ is the coefficient in the boundary term and is the secondary variable above. ω represents Dirichlet BC and is also called as *primary variable*. ω denotes the unknown function u, at boundaries. Whenever u=0 (from (2) $u = 0$ at $x = 0$), set $\omega = 0$. Applying this in (4) we get:

$$\left[\omega EA\frac{d\tilde{u}}{dx}\right]_L + \int_0^L \omega\ F\ dx = \int_0^L EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}\ dx \qquad \text{———} \qquad (4)$$

Equation (4) is called as *weak-form* of the PDE. The equation in (1) contains a second-order derivative term. This means that there is a requirement that u must be differentiable twice so that u is continuous (at all points x, $\partial^2 u/\partial x^2$ is real). In equation (3), this requirement is *weakened* in the sense that there are only first-order derivative terms. So, it is sufficient if u is differentiable once.