CS601: Software Development for Scientific Computing

Autumn 2021

Week9:

Unstructured Grids (Finite Element Method), Sparse Matrices

Course Progress..

- Last topic (two weeks ago..) - unstructured grids (with Delaunay triangulation)
  - Common in practical scenarios
  - Required to handle complex geometries
- Coming Next:
  - Computation on unstructured grids (with Finite Element Method (FEM))
  - Sparse Matrices

Finite Element Method

- Technique for solving PDEs
  - we have seen Finite Difference Method earlier
- Two step process:
  - Discretization:
    - local discretization over small, simple regions with triangles / quadrilaterals (finite elements) in 2D.
  - The equations for smaller regions are combined to form equivalent ones for larger regions
    - Conversion from strong form to weak form
    - Numerical solution of the weak form

From Strong Form to Weak Form

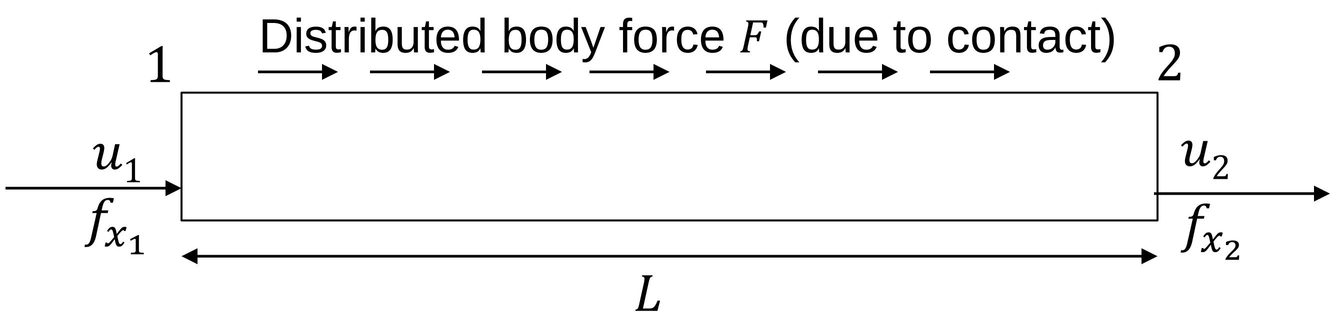
1. Principle of Virtual Work
2. Principle of Minimum Potential Energy
3. Method of weighted residuals (Galerkin, collocation, Least Squares methods etc.)
   - Galerkin is the most commonly used method.
     - Multiply by a weighting function
     - Integrate over the domain
     - Discretize the sum of contributions from each element
   - Apply the divergence theorem

Example : Rod Element

- Some background first..
  - Stress (σ) = Force per unit Area = P/A
    - P = Axial force (load applied along the length or $\perp$ to cross section),
    - A = area
  - Strain (ϵ) = Deformation in the direction of force applied = σ/E
  - Deformation = displacement of particles

Example : Rod Element

- Elastic rod with end points(nodes) 1 and 2 and length L
- Axial Force P and body force F
- Displacements u_1 and u_2 along horizontal direction at end nodes due to Axial Force P only (also called nodal displacements)

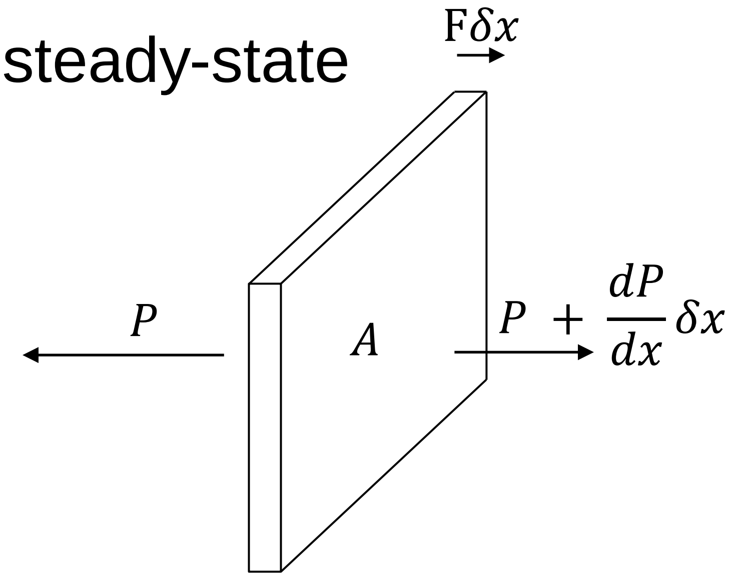


- Goal: to find displacements at various points in steady-state

Example : Rod Element

- P is then $= \sigma A = EA\epsilon = EA\frac{du}{dx}$
- Assuming a small strain and for steady-state (equilibrium)

$$\frac{dP}{dx} + F = 0$$



- Therefore, the equation to be solved:

$$EA\frac{d^2u}{dx^2} + F = 0 \qquad (1)$$

Example : Rod Element

- As per the FEM technique, continuous variable u in:

$$EA\frac{d^2u}{dx^2} + F = 0$$

is approximated by $\tilde{u}$ in terms of its nodal displacements u_i and u_j through shape/weight functions N_1, N_2 :

$$\tilde{u} = N_1u_1 + N_2u_2$$

or

$$\tilde{u} = [N_1 \quad N_2]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\boldsymbol{N}]\{\boldsymbol{u}\}$$

where, $N_1 = 1 - x/L$, $N_2 = x/L$ are simple linear functions

Why are N_1 and N_2 defined as the way they are?

Example : Rod Element

- Substituting: $EA\dfrac{d^2}{dx^2}\begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + F = \mathcal{R}$ ——— (2)

  where,

  $\mathcal{R}$ is a measure of error in approximation called residual.

- *We have replaced the original differential equation (1) in terms of nodal values in (2).*
  - *strong form to weak form*
- Problem is now reduced to finding good values of $\begin{matrix} u_1 \\ u_2 \end{matrix}$ to minimize $\mathcal{R}$

Example : Rod Element – Galerkin Method

1. Multiply / weight the residual in (2) by each shape function $[N_1 \quad N_2]$
2. Integrate over the domain and equate to zero.

$$\int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} EA \frac{d^2}{dx^2} [N_1 \quad N_2] \, dx \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} F dx = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

- Recall: our $N_1 = 1 - x/L$, $N_2 = x/L$ are simple *linear* functions (*piecewise linear functions*). So, double differentiation in $d^2/dx^2[N_1 \quad N_2]$ would make them vanish.

Example : Rod Element – Galerkin Method

- To overcome, we apply Green's theorem (integration by parts)

$$\int \; N_i \frac{\partial^2 N_j}{\partial x^2} dx = -\int \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx + \text{boundary terms}$$

- Boundary terms are ignored (only for Dirichlet cond.) to yield (from eqn. (3)):

$$-EA\int_0^L \begin{bmatrix} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} \end{bmatrix} dx \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} F dx = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

Example : Rod Element – Galerkin Method

- Evaluating integrals:

$$-EA\int_0^L \begin{bmatrix} \frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} \end{bmatrix} dx \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \int_0^L \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} F dx = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4)$$

$$-\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + FL\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{x_2} \end{bmatrix} \quad (5)$$

Here, the total force FL is shared equally among two nodes i and j

Example : Rod Element – Stiffness matrix

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_{x_1} \\ f_{x_2} \end{bmatrix} \qquad \text{——— (5)}$$

Writing (5) in matrix notation:

$$[\boldsymbol{k_m}]\{\boldsymbol{u}\} = \{\boldsymbol{f}\}$$

Where,

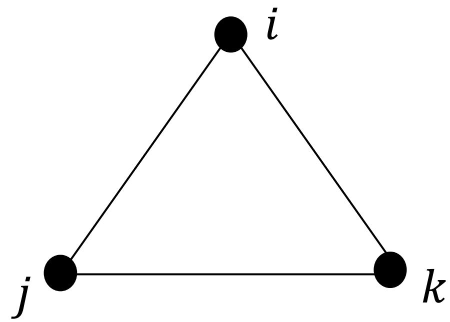
$[\boldsymbol{k_m}]$ = element stiffness matrix

$\{\boldsymbol{u}\}$ = element nodal displacements

$\{\boldsymbol{f}\}$ = element nodal forces vector

Triangle Elements?

- Start with triangle elements $i\ j, k$ for all triangles



- End with a system of linear equations consisting of global stiffness matrix K, the unknown coefficients vector a of the solution approximation, and Force vector F:

$$Ka = F$$

- The calculation of K and F is performed by looping over each element and sending the contributions from each element to the proper entry in K and F.

Triangle Elements?

- Common to find a resulting stiffness matrix that is:
  - Sparse
  - Symmetric
  - Positive-Definite
- In $Ka = F$, F is sometimes an integral. This is computed using numerical integration method such as Gaussian Quadrature.

FEM – Further Reading

- MIT OCW (2D Diffusion problem with triangle elements)
- IIT Madras NPTEL Lectures (Introduction to FEM, 1D rod problem. Series of lectures starting from this one.)
- Youtube video on overview of FEM (great animation and commentary)