Notes for the FEM class on 7/10/2021

Prof. Amar started with a brief summary of the FEM approach that we have been following for 2D problems: start with governing equations which are in the strong-form. Next, use integration by parts to reduce the degree of the degree of the equation e.g. to substituting second-order with first-order derivative terms. In case of 2D problems, for some of the terms obtained in the previous steps, apply the Gauss-divergence theorem to convert the domain or surface integral to boundary integral to obtain the weak-form equation. Once the weak-form equation is obtained, write elemental equations. To do this, first discretize the domain into N-sided elements, where each element is assumed to have N nodes. After elemental equations for all the elements of the domain are written, perform assembly to construct the global stiffness matrix, boundary term vector, and force vector. Note that the elements of the Global stiffness matrix and force vector involve computation of integrals over limits in the physical domain. Analytically solving the integrals is too difficult or not possible often. Hence, to numerically solve the integrals, Gauss-Quadrature method is used. The details of this method were discussed in the last class.

In today's class Prof. Amar continued with the weak-form equation obtained in the last class for 2D steady state heat diffusion problem to write elemental equations with n-sided elements in general. The equations written had the symbolic weight function (ω) for a general n-sided element. These symbolic weight functions were replaced by the shape functions (N_i)s that were derived for triangular elements. Thus, we obtained 3 equations per element consisting of 3 bilinear integral terms on the LHS. In other words, the elemental stiffness matrix was 3x3.

Prof. Amar then explained the assembly of such 3x3 matrices and showed the constraints / requirements to be met while ensuring the continuity of the solution at nodes common to elements.

We begin with the weak-form equation obtained in the last class:

$$\int_\Omega \mathrm{K}\left(\frac{\partial\omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}+\frac{\partial\omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right)\mathrm{d}\Omega = \int_\Gamma \mathrm{K}\omega\left[\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\hat{n}_y+\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\hat{n}_x\right]\mathrm{d}\Gamma + \int_\Omega \omega f\,\mathrm{d}\Omega \quad \text{—— (1)}$$

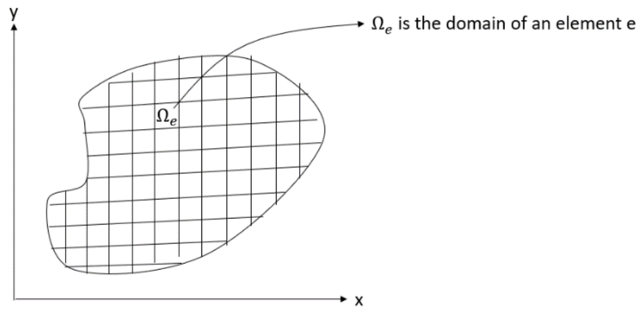
Stiffness matrix

Boundary condition term (vector) contains coefficient of weight function

$$\mathrm{K}\left[\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\hat{n}_y+\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\hat{n}_x\right]$$

Force vector

Before we write the elemental equations, we need to discretize the domain. The picture illustrates an example:



Writing the weak-form equation for the element's domain Ω_e

$$\int_{\Omega^e} K\left(\frac{\partial\omega}{\partial x}\frac{\partial T^e}{\partial x}+\frac{\partial\omega}{\partial y}\frac{\partial T^e}{\partial y}\right)d\Omega^e = \int_{\Gamma^e} K\omega\left[\frac{\partial T^e}{\partial y}\hat{n}_y+\frac{\partial T^e}{\partial x}\hat{n}_x\right]d\Gamma^e+\int_{\Omega^e}\omega f d\Omega^e$$

Where, T^e denotes the approximate solution (temperature) within the element's domain, and Γ^e denotes element's boundary.

The above equation can also be written as:

$$\int_{\Omega^e} K\left(\frac{\partial\omega}{\partial x}\frac{\partial T^e}{\partial x}+\frac{\partial\omega}{\partial y}\frac{\partial T^e}{\partial y}\right)d\Omega^e = \int_{\Gamma^e}\omega\hat{q}^n d\Gamma^e+\int_{\Omega^e}\omega f d\Omega^e \quad \text{——} \quad (2)$$

Where, $\hat{q}^n$ denotes the flux for the element. Note that this flux is different from the flux q^n specified for the entire domain.

Also, T^e can be written as: $\sum_{i=1}^{n} N_i T_i^e$

- where, T_i^e is the temperature at the ith node of the element (Similar to what we did in class on 5/10 when we approximated the displacement, $u(x)$, between the nodes of an element in terms of nodal displacements using linear functions: $\tilde{u}(x) = N_1(x)u_1 + N_2(x)u_2$)
- and N_i are shape functions (now of two variabes $N_i(x,y)$).

As per the Galerkin approach, ω can be replaced with N_i. Rewriting (2):

$$\int_{\Omega^e} K\left(\frac{\partial N_i}{\partial x}\frac{\partial T^e}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial T^e}{\partial y}\right)d\Omega^e = \int_{\Gamma^e} N_i\hat{q}^n d\Gamma^e+\int_{\Omega^e} N_i f d\Omega^e$$

Substituting for $T^e = \sum_{i=1}^{n} N_i T_i^e$ the above equation:

$$\int_{\Omega^e} K\left(\frac{\partial N_i}{\partial x}\frac{\partial}{\partial x}\sum_{j=1}^{n} N_j T_j^e+\frac{\partial N_i}{\partial y}\frac{\partial}{\partial y}\sum_{j=1}^{n} N_j T_j^e\right)d\Omega^e = \int_{\Gamma^e} N_i\hat{q}^n d\Gamma^e+\int_{\Omega^e} N_i f d\Omega^e$$

The summation symbol can be omitted (as per the conventional notations) to represent the above equation as:

$$\int_{\Omega^e} K\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)T_j^e d\Omega^e \;=\int_{\Gamma^e} N_i\hat{q}^n d\Gamma^e+\int_{\Omega^e} N_i f d\Omega^e$$

The T_j^e term can be factored out because the nodal temperature is assumed to be a constant (unknown).

The above equation is sometimes compactly written as:

$$K_{ij}T_j^e = q_i + f_i \tag{3}$$

$$\text{where, } K_{ij} = \int_{\Omega^e} \mathrm{K}\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\,d\Omega^e$$

Equation (3) needs to be written for all nodes of an element and the exercise needs to be repeated for all elements of the domain. The matrices obtained w.r.t. an element need to be assembled afterwards.

For an n-sided element:

$$T \;=\; N_1T_1 + N_2T_2 + \ldots + N_nT_n$$

Hence,

$$\frac{\partial T}{\partial x} = \frac{\partial N_1}{\partial x}T_1 + \frac{\partial N_2}{\partial x}T_2 + \ldots + \frac{\partial N_n}{\partial x}T_n$$
$$\frac{\partial T}{\partial y} = \frac{\partial N_1}{\partial y}T_1 + \frac{\partial N_2}{\partial y}T_2 + \ldots + \frac{\partial N_n}{\partial y}T_n$$

Rewriting in Ax=b form:

$$\begin{bmatrix}\frac{\partial T}{\partial x}\\ \frac{\partial T}{\partial y}\end{bmatrix} = \begin{bmatrix}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\ldots & \frac{\partial N_n}{\partial x}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y}\ldots & \frac{\partial N_n}{\partial y}\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ \vdots\\ T_n\end{bmatrix}$$

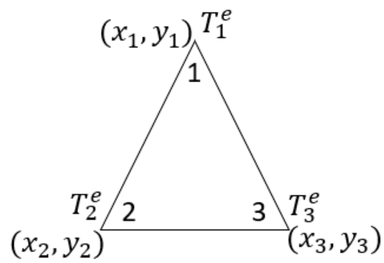
Call this matrix as B

Define a matrix C as: $\begin{bmatrix}\mathrm{K} & 0\\ 0 & \mathrm{K}\end{bmatrix}$ then,

$$K_{ij} = \int_{\Omega^e} B^T C B\, d\Omega^e$$

where, B^T denotes transpose of the matrix B.

Deriving shape functions for a triangular element, n=3 (n=3 sides):



$$T = C_1 + C_2 x + C_3 y$$
$$T_1^e = C_1 + C_2 x_1 + C_3 y_1$$
$$T_2^e = C_1 + C_2 x_2 + C_3 y_2$$
$$T_3^e = C_1 + C_2 x_3 + C_3 y_3$$

Solving for C_1 , C_2, and C_3 (in terms of x and y), and substituting in $T = C_1 + C_2 x + C_3 y$ we get

T = ( ) T_1^e + ( ) T_2^e + ( ) T_3^e

↑ ↑ ↑

Denotes: N_1 N_2 N_3

$$N_1 = \frac{1}{2A^e}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

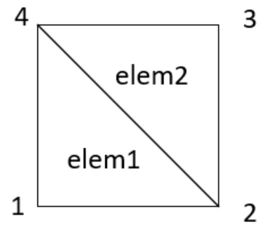
$$N_2 = \frac{1}{2A^e}[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$N_3 = \frac{1}{2A^e}[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

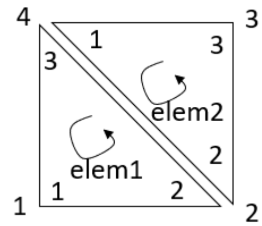
Where, A^e is the area of the triangular element.

For the above N_i's, the elements of the B matrix are constants (i.e. not a function of x or y because N_is are linear functions of x or y and the partial derivatives of N_is give out constants.) Such a constant matrix is called in the literature as "Constant strain triangular element".

Once such matrix is obtained for all elements, we do the assembly. While doing assembly, first number the nodes in anti-clockwise in local node numbering as shown:



Global numbering of nodes



Local numbering of nodes

To maintain continuity of the solution obtained (i.e. temperature at nodes common to elements must be same and the flux should be zero on the common edges – there are two common nodes 2 and 4):

$T_1 = T_1^{e_1}$ (meaning temperature at node 1 in global numbering $=$ temperature at node 1 of element 1)

$$T_4 = T_3^{e_2}$$
$$T_2 = T_2^{e_1} = T_2^{e_2}$$
$$T_4 = T_3^{e_1} = T_1^{e_2}$$

Regrading flux term (q_i in Eqn. (3)):
At node number 2 (global numbering):
The outflux of node 2 in element 1 + outflux of node 2 in element 2 = 0

$$\int_{\Gamma^{2-3}} N_2 q_{2-3} d\Gamma^{2-3} + \int_{\Gamma^{1-2}} N_2 q_{1-2} d\Gamma^{1-2}$$

Similarly, at node number 4 (global numbering):
The outflux of node 3 in element 1 + outflux of node 1 in element 2 = 0

$$\int_{\Gamma^{2-3}} N_3 q_{2-3} d\Gamma^{2-3} + \int_{\Gamma^{1-2}} N_1 q_{1-2} d\Gamma^{1-2}$$