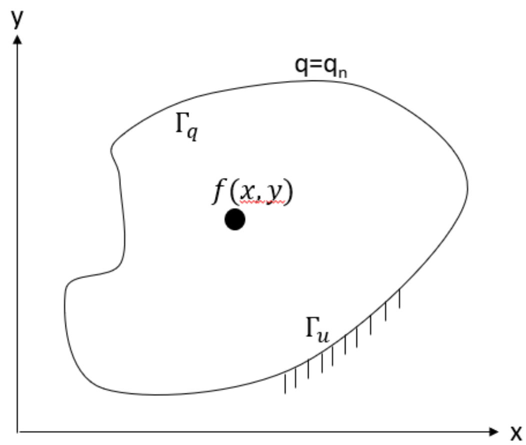
Notes for the FEM class on 6/10/2021

Prof. Amar started with a recap explaining the fundamental difference (Pros and Cons of) between Finite Difference Method (FDM) and Finite Element (FEM): FDMs are used when the domain is well structured. With Neumann boundary conditions specified, FDM approach can be complicated. FEM is more popular because it can handle unstructured/complicated domains in addition to structured domains. In practical scenarios, the domains are often unstructured, and FEM can yield better solution accuracy. Prof. Amar also briefly touched upon the concepts of primary and secondary variables (Primary variables e.g. are displacements, u, as seen in the Rod problem. Secondary variables are associated with Neumann Boundary Conditions (B.C.) e.g. when $F = EA\frac{\partial u}{\partial x}$ the computation of us are for secondary variables. We say that Dirichlet B.C. are imposed on primary variables and Neumann B.C. are imposed on secondary variables.). Fact: For a *well-posed* problem, you must have Dirichlet B.C. specified at least one of the nodes.

The main topic of today's class was 2D steady state diffusion problem. Prof. Amar started with the strong-form of the steady state heat diffusion equation and derived the weak-form. The problem is as illustrated below:



- Γ_u= Part of the boundary, where boundary conditions are specified. Here, $T_{\Gamma_u} = \tilde{T}$, meaning the temperature is known at boundary Γ_u.
- Γ_q= Part of the boundary, where *flux* is specified.
- Flux = q = q_n =incoming or outgoing heat energy
- $f(x,y)$ = Heat source
- Note: $\Gamma = \Gamma_u + \Gamma_q$, represents entire boundary.

Problem: *find the temperature T at any point (x,y) over the domain.*

The steady state heat diffusion equation is given by:

$$\mathrm{K}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + f(x,y) = 0 \quad (1)$$

When $f(x,y) = 0$, the above equation becomes $\mathrm{K}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = 0$ and is also called a Laplace's equation.

We also know that at some boundary Γ_u :

$$T_{\Gamma_u} = \tilde{T} \quad (2)$$

And the Neumann B.C. along boundary Γ_q ($\hat{n}_x$ and $\hat{n}_y$ are unit vectors along x and y direc resp.)

$$\mathrm{K}\left(\frac{\partial T}{\partial x}\hat{n}_x + \frac{\partial T}{\partial y}\hat{n}_y\right) = q_n \quad (3)$$

(1), (2), and (3) represent the strong-form of the steady-state 2D heat diffusion problem.
The first step in the FEM approach is to transform the strong-form to weak-form. This is done by integrating the product of the weight function and the residual over the domain and equating to zero i.e.

$$\int_{\Omega} \omega\, R = 0$$

$$= \quad \int_{\Omega} \omega \left(\mathrm{K}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + f(x,y) \right) = 0 \quad \text{—— (4)}$$

Road to obtaining weak-form:

We know that:

$$\frac{\partial}{\partial x}\left[\omega \mathrm{K} \frac{\partial T}{\partial x}\right] = \mathrm{K}\frac{\partial \omega}{\partial x}\frac{\partial T}{\partial x} + \omega \mathrm{K}\, \frac{\partial^2 \mathrm{T}}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left[\omega \mathrm{K} \frac{\partial T}{\partial y}\right] = \mathrm{K}\frac{\partial \omega}{\partial y}\frac{\partial T}{\partial y} + \omega \mathrm{K}\, \frac{\partial^2 \mathrm{T}}{\partial y^2}$$

Substituting for the second-order partial derivative term in (4):

$$-\int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial \mathrm{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\, d\Omega + \int_{\Omega} \frac{\partial}{\partial \mathrm{x}}\left(\omega \mathrm{K}\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right) d\Omega - \int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial \mathrm{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\, d\Omega + \int_{\Omega} \frac{\partial}{\partial \mathrm{y}}\left(\omega \mathrm{K}\frac{\partial \mathrm{T}}{\partial \mathrm{y}}\right) d\Omega + \int_{\Omega} \omega f\, d\Omega = 0 \quad \text{—— (5)}$$

Background (vector calculus):

- A function that takes in e.g. two variables (x,y in 2D) and outputs *one value* given by $f(x,y)$ is called a *scalar-valued function*.
- A function that takes in e.g. two variables (x,y in 2D) and outputs *a vector* in (x,y), i.e. the value of $f(x,y)$ is a vector, is called a *vector-valued function*.
- Suppose we define a function that outputs a vector $\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$, where the vector components are partial derivatives of the function f, we call such a vector-valued function the *gradient* or ∇ (nabla). $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\hat{n}_x + \frac{\partial f}{\partial y}\hat{n}_y$, where $\hat{n}_x$ and $\hat{n}_y$ are unit vectors along x and y direction resp.
- Divergence of a vector field $\nabla \,.\, (f_x, f_y)$ is a scalar-valued function $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}$
- Let a be a vector field. Then, the Gauss divergence theorem relates the surface integral (2D) to boundary/contour/line integral (1D) through the divergence operator:

$$\int_{\Omega} \nabla . a \, d\Omega = \int_{\Gamma} a_i \, \hat{n}_i d\Gamma$$

Applying Gauss divergence theorem in (5):

$$-\int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial x}\frac{\partial T}{\partial x} \, d\Omega$$

$$+\int_{\Omega} \frac{\partial}{\partial x}\left(\omega \mathrm{K}\frac{\partial T}{\partial x}\right) d\Omega \quad \leftarrow \quad \int_{\Gamma} \mathrm{K}\omega \frac{\partial T}{\partial x} \, \hat{n}_x d\Gamma$$

$$-\int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial y}\frac{\partial T}{\partial y} \, d\Omega$$

$$+\int_{\Omega} \frac{\partial}{\partial y}\left(\omega \mathrm{K}\frac{\partial T}{\partial y}\right) d\Omega \quad \leftarrow \quad \int_{\Gamma} \mathrm{K}\omega \frac{\partial T}{\partial y} \, \hat{n}_y d\Gamma$$

$$+\int_{\Omega} \omega f d\Omega = 0$$

$$= \int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial x}\frac{\partial T}{\partial x} \, d\Omega + \int_{\Omega} \mathrm{K}\frac{\partial \omega}{\partial y}\frac{\partial T}{\partial y} \, d\Omega = \int_{\Gamma} \mathrm{K}\omega \left[\frac{\partial T}{\partial y}\hat{n}_y + \frac{\partial T}{\partial x}\hat{n}_x\right] d\Gamma + \int_{\Omega} \omega f d\Omega$$

$$= \int_{\Omega} \mathrm{K}\left(\frac{\partial \omega}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial \omega}{\partial y}\frac{\partial T}{\partial y}\right) d\Omega = \int_{\Gamma} \mathrm{K}\omega \left[\frac{\partial T}{\partial y}\hat{n}_y + \frac{\partial T}{\partial x}\hat{n}_x\right] d\Gamma + \int_{\Omega} \omega f d\Omega \quad \text{—— (6)}$$

Stiffness matrix

Boundary condition term (vector) contains coefficient of weight function

$$\mathrm{K}\left[\frac{\partial T}{\partial y}\hat{n}_y + \frac{\partial T}{\partial x}\hat{n}_x\right]$$

Force vector

The stiffness matrix is identified by the bilinear term $\frac{\partial \omega}{\partial x}\frac{\partial T}{\partial x}$ or $\frac{\partial \omega}{\partial y}\frac{\partial T}{\partial y}$

(6) is the weak-form equation