Notes for the FEM class on 5/10/2021

The Ax=B form obtained in the last class (on 30/9/2021) involved computation of integrals over the domain (for computing the K matrix and for computing the body force vector term in the RHS). Sometimes, it is non-trivial to compute the integration. When E and A are functions of space variables (x), when higher-order polynomial shape functions are used to approximate the solution for better accuracy (e.g. with 3-node or 5-node elements we need higher-order polynomial functions.), we need to numerically compute the integral function. Also, the domain always consists of more than one element. In this scenario, the elemental stiffness matrices from many constituent elements are to be combined to form a global stiffness matrix.

Prof. Amar explained in this class the technique, Gauss-Quadrature, used commonly in FEM to numerically integrate functions appearing as terms in the Ax=B form. Prof. Amar also explained how the Gauss-Quadrature technique is used to map the physical domain of the 1D rod element to the natural domain required by the Gauss-Quadrature before performing numerical integration. Finally, Prof. Amar concluded the discussion with constructing a global stiffness matrix for the rod consisting of *two* elements.

Gauss-Quadrature has different rules such as 1-point, 2-point, ..., N-point etc. Depending on the polynomial appearing as the integral function, you use different rules. *Fact: using N-point quadrature rule, you can integrate (2N-1)-degree polynomial correctly.*

The basic requirement for the Gauss-Quadrature rules to be applied is that the integral limits range from -1 to 1. However, the limits in the integral terms appearing in the elemental equations in Ax=B form range from x_a to x_b (x_a and x_b are coordinates in the *physical domain* of the rod element. The range -1 to 1 is referred to as the *natural domain*.). Hence, the integral functions need to be *mapped* from physical domain to the natural domain. E.g.

$$K_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx} \frac{dN_j}{dx} dx = \int_{x_a}^{x_b} F(x)dx \qquad \Rightarrow \qquad K_{ij} = \int_{-1}^{1} \tilde{F}(\xi)d\xi$$

$\int_{-1}^{1} \tilde{F}(\xi)d\xi$ is numerically computed as summation:

$\int_{-1}^{1} \tilde{F}(\xi)d\xi = \sum_{i=1}^{N} w_i \tilde{F}(\xi_i)$, where w_i is the quadrature weight of the ith quadrature point and ξ_i is the corresponding location. E.g.

In two-point quadrature $w_1 \tilde{F}(\xi_1) + w_2 \tilde{F}(\xi_2)$, $w_1 = w_2 = 1, \xi_1 = -\frac{1}{\sqrt{3}}, \xi_2 = \frac{1}{\sqrt{3}}$.

In one-point quadrature, $w_1 \tilde{F}(\xi_1), w_1 = 2, \xi_1 = 0$

Earlier, the displacement at x, a point between the nodes, was written in terms of the nodal displacements using linear functions N_1 and N_2 as follows:

$$\tilde{u}(x) = N_1(x)u_1 + N_2(x)u_2$$

where u_1 and u_2 are nodal displacements at nodes 1 and 2 resp.

Now, we would need the equivalent of:
$$\tilde{u}(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$$

When $x = x_a, \xi = -1$ and $x = x_b, \xi = 1$.

Therefore, $x = x_a \frac{1-\xi}{2} + x_b \frac{1+\xi}{2}$. So, we can write x as:

$x = x_a N_1(\xi) + x_b N_2(\xi)$, where $N_1 = \frac{1-\xi}{2}$ and $N_2 = \frac{1+\xi}{2}$ can be seen as linear functions of ξ.

In *isoparametric* FEM, we use the same functions N_1 and N_2 to approximate the solution $\tilde{u}(\xi)$ as:
$$\tilde{u}(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$$

Now, we need to express the N_i and N_j (which are linear functions of x) in $K_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx} \frac{dN_j}{dx} dx$ using $N_1(\xi)$ and $N_2(\xi)$. So,

$\frac{dN_i}{dx}$ needs to be converted in terms of $\frac{dN_i}{d\xi}$ and $\frac{d\xi}{dx}$ (chain rule).

We know that: $x = x_a N_1(\xi) + x_b N_2(\xi)$. OR $x = x_1 N_1 + x_2 N_2$ ($x_1 = x_a$ and $x_2 = x_b$ are node 1 and node 2 coordinates resp.) OR $x = \sum_{k=1}^{2} x_k N_k$

$\frac{dx}{d\xi} = x_1 \frac{d}{d\xi} N_1 + x_2 \frac{d}{d\xi} N_2 = \sum_{k=1}^{2} x_k \frac{d}{d\xi} N_k$,

$\frac{dx}{d\xi} = J$, where J is called the Jacobian matrix (because the domain is 1D, we have a scalar value (only one component) for the Jacobian matrix).

$$K_{ij} = \int_{x_a}^{x_b} EA \frac{dN_i}{dx} \frac{dN_j}{dx} dx = \int_{-1}^{1} EA \frac{dN_i}{d\xi} \frac{d\xi}{dx} \frac{dN_j}{d\xi} \frac{d\xi}{dx} \left(\sum_{k=1}^{2} x_k \frac{d}{d\xi} N_k \right) d\xi =$$
$$\int_{-1}^{1} EA \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} [J]^{-1}[J]^{-1}[J] d\xi =$$
$$\int_{-1}^{1} EA \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} [J]^{-1} d\xi$$

The integral function $EA \frac{dN_i}{d\xi} \frac{dN_j}{d\xi} [J]^{-1}$ above can be represented as $\tilde{F}(\xi)$

Using 1-point quadrature ($w_1 = 2, \xi_1 = 0$) and substituting for $\tilde{F}(0)$:

$$\frac{dx}{d\xi} = x_1 \frac{d}{d\xi} N_1 + x_2 \frac{d}{d\xi} N_2 = x_1 \frac{d}{d\xi}\left(\frac{1-\xi}{2}\right) + x_2 \frac{d}{d\xi}\left(\frac{1+\xi}{2}\right) = \frac{-x_1}{2} + \frac{x_2}{2} = \frac{x_2 - x_1}{2} = \frac{L}{2}$$

$$K_{11} = \int_{-1}^{1} EA \frac{dN_1}{d\xi}\frac{dN_1}{d\xi}[J]^{-1}d\xi = \frac{EA}{L}$$

$$K_{12} = \int_{-1}^{1} EA \frac{dN_1}{d\xi}\frac{dN_2}{d\xi}[J]^{-1}d\xi = -\frac{EA}{L}$$
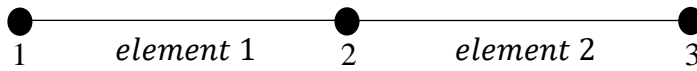
$$K_{21} = \int_{-1}^{1} EA \frac{dN_2}{d\xi}\frac{dN_1}{d\xi}[J]^{-1}d\xi = -\frac{EA}{L}$$

$$K_{22} = \int_{-1}^{1} EA \frac{dN_2}{d\xi}\frac{dN_2}{d\xi}[J]^{-1}d\xi = \frac{EA}{L}$$

Thus, the elemental stiffness matrix is (when E and A are constants):

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now consider the rod with two elements. There are 3 nodes as shown below for the two elements:

For the 3 nodes, there are three degrees of freedom (i.e. 3 displacement values, each corresponding to a node). Since, there are 3 displacement values i.e. u^s , there would be 3 components to the x vector in the Ax=B form (Recall: x vector denotes the displacements). So, the *global stiffness matrix* combining the elemental matrices of elements 1 and 2 would be 3x3:

Node:	1	2	3

Node:

$\dfrac{A_1 E_1}{L_1}$	$-\dfrac{A_1 E_1}{L_1}$	0
$-\dfrac{A_1 E_1}{L_1}$	$\dfrac{A_1 E_1}{L_1} + \dfrac{A_2 E_2}{L_2}$	$-\dfrac{A_2 E_2}{L_2}$
0	$-\dfrac{A_2 E_2}{L_2}$	$\dfrac{A_2 E_2}{L_2}$

(Node 1, 2, 3 correspond to the rows.)

, where $\frac{A_1 E_1}{L_1}$ comes from elemental matrix for element 1 and $\frac{A_2 E_2}{L_2}$ comes from elemental matrix for element 2. Note that the particular order of numbering nodes is important and dictates the values of the entries in the global stiffness matrix. Based on the numbering followed the resultant stiffness

matrix can be sparse (lot of zeros) or non-sparse. For a 1D problem with many elements and following the node numbering order as shown above, we get a tridiagonal matrix.

Note: If you understand FEM by this time, then it is because of the excellent guest lectures by Prof. Amar Gaonkar.