Notes for the FEM class on 30/9/2021

Prof. Amar started with an explanation of why it helps to consider 3D objects such as a rod as a 1D object for the purpose of modeling to understand certain processes/phenomenon. He started with the specific PDE formula used (`Eq.1 on slide 7`), as a mathematical model, to understand the displacement of particles at various points in a rod when the rod is subjected to certain kinds of forces.

$$EA\frac{d^2u}{dx^2} + F = 0$$

, where E is often a constant, A is area of the cross section (unit area)

This formula, he explained, is in "Strong-form", has a second-order derivative term, and hence, requires a quadratic polynomial function to accurately represent it whenever such representation exists. He then explained that a series of functions (linear in this case), called shape functions or weight functions, are used to approximate the solution, $\tilde{u}$, mentioned in the PDE formula. Being approximate in nature, $\tilde{u}$ has an associated error or *Residual* R. Therefore, we would have:

$$EA\frac{d^2\tilde{u}}{dx^2} + F = R$$

He then derived what is called as the "Weak-form" formulation for this approximate function for the 1D problem using FEM employing Galerkin approach / weighted residuals. In general, the Galerkin approach would integrate the weighted residual and equate it to zero i.e.

$$\int \omega\, R = 0$$

, where ω are the weight functions or shape functions and R is the residual. For the 1D problem, this general formula can be written as (substituting for R):

$$\int \omega\, (EA\frac{d^2\,\tilde{u}}{dx^2} + F) = 0$$

$=\int_\Omega\ \omega\, EA\frac{d^2\,\tilde{u}}{dx^2}\ d\Omega\ + \int_\Omega\ \omega\, F\, d\Omega = 0$, where $\int_\Omega$ denotes integral over the domain Ω

$\omega\ \frac{d^2\,\tilde{u}}{dx^2}$ in the first term above, $\int_\Omega\ \omega\, EA\frac{d^2\,\tilde{u}}{dx^2}\ d\Omega$, can be substituted using the information:

$$\frac{d}{dx}\left[\omega\frac{d\tilde{u}}{dx}\right] = \frac{d\omega}{dx}\frac{d\tilde{u}}{dx} + \omega\ \frac{d^2\,\tilde{u}}{dx^2}$$

Substituting:

$$\int_\Omega \frac{d}{dx}(\omega EA\frac{d\tilde{u}}{dx})\, d\Omega\ - \int_\Omega\ EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}\ d\Omega\ + \int \omega\, F\, d\Omega\ = 0$$

$$= \left[\omega EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega\ \omega\, F\, d\Omega = \int_\Omega\ EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}\ d\Omega$$

In the above general formula that Prof. Amar derived, he explained what each term intuitively meant: Stiffness matrix, body force vector, boundary condition coefficients.

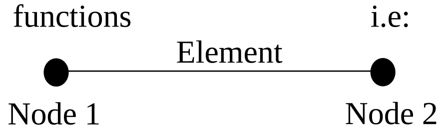
$$\left[\omega EA\frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega \omega\, F\, d\Omega = \int_\Omega EA\frac{d\omega}{dx}\frac{d\tilde{u}}{dx}\, d\Omega$$

Boundary condition coefficient vector — Body force vector — Stiffness matrix

The above is the "Weak-form" formula. This formula is preferred over "Strong-form" because of lower order term (i.e. only the first-order derivative terms are present and second-order derivative term of the "Strong-form" formula is absent.) making it relatively easier to numerically compute.

Then, he took the example of 1D rod fixed at one end and subjected to body and axial forces. Examples of body forces are gravitational, electromagnetic etc. He considered the rod as a single element with the two endpoints serving as nodes. He explained how when given two points, one can fit/approximate a linear function connecting the points. He then went on to explain the origin of the weight functions associated with the two nodes of the rod element.

For a two-node element, where the elements have basis functions (or weight/shape functions) N_1 and N_2, the method of weighted residuals says that the displacement, u, at any point in between the nodes (note: nodes are located at end points of the element.) can be approximated using linear combination of the basis functions i.e:



$$\tilde{u} = N_1 u_1 + N_2 u_2$$

or written in terms of longitudinal displacement:

$$\tilde{u}(x) = N_1(x)u_1 + N_2(x)u_2$$

where u_1 and u_2 are displacements at the Nodes 1 and 2 resp. These are called nodal / elemental displacements. N_1 and N_2 (functions of x) can be derived to get:

$$N_1 = 1 - x/L$$
$$N_2 = x/L$$

Further, he also mentioned what Galerkin suggested: choose the weight functions appearing in the "weak-form" equation as the same as the ones derived for the nodes i.e. replace ω in the "weak-form" equation with N_1 and N_2:

So, the weak-form formula was rewritten for each element substituting for the weight functions to yield Stiffness matrix, Force vector (body force + boundary condition coefficients), and displacements.

Substituting for ω with N_1 and N_2. in the "weak-form" equation, we get the following two equations:

$$\left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_1\ F\ d\Omega = \int_\Omega EA \frac{dN_1}{dx}\frac{d}{dx}(N_1 u_1 + N_2 u_2\ )\ d\Omega$$

$$\left[N_2 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_2\ F\ d\Omega = \int_\Omega EA \frac{dN_2}{dx}\frac{d}{dx}(N_1 u_1 + N_2 u_2\ )\ d\Omega$$

Considering the first equation, rewriting, and expanding:

$$\left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_1\ F\ d\Omega = \int_\Omega EA \frac{dN_1}{dx}\frac{d}{dx}(N_1 u_1 + N_2 u_2\ )\ d\Omega \qquad \text{(LHS=RHS)}$$

⇨ $\int_\Omega EA \frac{dN_1}{dx}\frac{d}{dx}(N_1 u_1 + N_2 u_2\ )\ d\Omega\ = \left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_1\ F\ d\Omega$ (RHS=LHS)

⇨ $\int_\Omega \mathrm{EA}\ \frac{dN_1}{dx}\frac{dN_1}{dx}\mathrm{u}_1\ d\Omega\ +\ \int_\Omega \mathrm{EA}\ \frac{dN_1}{dx}\frac{dN_2}{dx}\mathrm{u}_2\ d\Omega = \left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_1\ F\ d\Omega$

Similarly considering the second equation, rewriting, and expanding:

$$\int_\Omega \mathrm{EA}\ \frac{dN_2}{dx}\frac{dN_1}{dx}\mathrm{u}_1\ d\Omega\ +\ \int_\Omega \mathrm{EA}\ \frac{dN_2}{dx}\frac{dN_2}{dx}\mathrm{u}_2\ d\Omega = \left[N_2 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_\Omega N_2\ F\ d\Omega$$

Using shorter notation on the LHS for the two equations expanded:

$$K_{11}u_1 + K_{12}u_2 = \left[N_1 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_0^L N_1\ F\ dx \qquad \text{———— (1)}$$

$$K_{21}u_1 + K_{22}u_2 = \left[N_2 EA \frac{d\tilde{u}}{dx}\right]_0^L + \int_0^L N_2\ F\ dx \qquad \text{———— (2)}$$

Where, $K_{ij} = \int_0^L EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx$ and Ω ranges from 0 to L (dΩ becomes dx because of the domain is 1D).

The Equations 1 and 2 can be expressed in Ax=B form (A is matrix, x is vector, and B is a vector) as:

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \left[N_1 EA \dfrac{d\tilde{u}}{dx} \right]_0^L + \displaystyle\int_0^L N_1\, F\, dx \\ \left[N_2 EA \dfrac{d\tilde{u}}{dx} \right]_0^L + \displaystyle\int_0^L N_2\, F\, dx \end{bmatrix}$$

Stiffness matrix

displacements

Force vector